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^{3}He–HeII Mixtures in the Static Approximation

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We have applied, for the first time, the so-called static approximation (SA) on the ^{3}He–HeII mixtures to calculate their thermodynamic properties under different conditions. For that, we have first obtained the so-called self-consistent *long-range-equation*. The essence of this approximation is to replace the true quantum-mechanical spectrum of the local-field operator with a distribution around the expectation value of the local-field operator. The distribution function of the fermionic particles has been found to be almost a step function up to 0.1 K. The chemical potential, pressure, and the mean internal energy per unit volume have been calculated in this temperature range with different values for the volume differential coefficient α and the effective mass m_3^*. We have found that the thermodynamic properties of this system are temperature independent.

KEY WORDS: ^{3}He–HeII mixtures; static approximation.

1. INTRODUCTION

We have applied, for the first time, the so-called static approximation (SA) on the ^{3}He–HeII mixtures to calculate their thermodynamic properties as a function of temperature, density, and the effective mass of ^{3}He atoms. For that, we have first obtained the so-called self-consistent *long-range-equation*.

The alternative approximation to this approach is the static fluctuation approximation (SFA) (Al-Sugheir *et al.*, 2001; Al-Sugheir and Ghassib, 2002; Nigmatullin *et al.*, 2000a,b,c). The essence of the two approximations is essentially the same. In the SFA the square of the quadratic fluctuation of the local-field operator is replaced with its mean value, while in the SA the square of the local-field operator is replaced with its mean value. In both approximations, the true quantum-mechanical spectrum of the local-field operator is replaced with a distribution around the expectation value of the local-field operator. The new form of the SFA that is called SA makes the numerical calculations simpler. In general both approaches are relatively simple compared to the other many-body approaches.

The SFA has been used to study different physical systems such as the classical two-dimensional Ising model (Nigmatullin *et al.*, 2000a); the proton model of

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ferroelectrics with tunneling (Nigmatullin *et al.*, 2000b); the one-, two- and three-dimensional Ising model (Nigmatullin *et al.*, 2000c); liquid helium-4 (Al-Sugheir *et al.*, 2001); and liquid helium-3 (Al-Sugheir and Ghassib, 2002).

This paper considers an extended uniform normal liquid helium-3 system of N_3 atoms, each of mass m_3, occupying a volume Ω_3. This liquid is embedded in liquid HeII background of N_4 atoms, occupying a volume Ω_4. It should be noted that this mixture of ^{3}He–HeII, at temperatures less than the Fermi degeneracy temperature ($\sim$0.1 K at zero pressure), is a dilutely interacting neutral many-fermionic system. This is due to the negligible number density of Bose-type excitations (phonons and rotons) under these circumstances, and the dominance of the ^{3}He quasiparticles excitations in the mixture (Wilks, 1967).

These ^{3}He–HeII mixtures are of interest for a number of reasons. For example, they are widely used as refrigerants for cooling purposes up to the mK range (Kelly, 1973). They also play a central role as "theoretical laboratories" for investigating the validity of many-body formalisms insofar as these systems are not highly dense and strongly interacting such as liquid ^{3}He. The Fermi-liquid behavior has been fully vindicated both theoretically (Edward and Pettersen, 1992; Emery, 1966, 1967) and experimentally (Ebner and Edwards, 1970; König *et al.*, 1994, 1995; König and Pobell, 1994; Tuoriniemi *et al.*, 2002).

The basic phenomenological theory for dilute ^{3}He–HeII mixtures is that of Landau and Pomaranchuk (Edwards and Pettersen, 1992; Owers-Bradley, 1997). According to this theory, the ^{3}He quasiparticles interact with each other so weakly that the system behaves as an almost ideal Fermi gas. Microscopically, there have been the variational track, including the correlation-basic-function theory (Feenberg, 1969), and the perturbative track including the Galitskii-Migdal-Feynman theory (Bishop *et al.*, 1976; Ghassib *et al.*, 1976).

The synopsis of the rest of this paper is as follows. In Section 2 the effective interaction for a dilute ^{3}He–HeII is illustrated. In Section 3 the SA formalism is applied to our system, and in Section 4 a closed set of nonlinear integral equations for the ^{3}He–HeII mixtures is derived. Results and discussion are presented in Section 5, while concluding remarks are given in Section 6.

2. THE EFFECTIVE INTERACTION FOR DILUTE ^{3}He–HeII MIXTURES

The effective interaction in configuration space between two ^{3}He quasiparticles embedded in HeII is the sum of three physical effects. The first effect is the direct ^{3}He–^{3}He interaction, V_{33}. We have used one of the most highly acclaimed interatomic helium potentials, the so-called HFDHE2 potential (Aziz *et al.*, 1979; Janzen and Aziz, 1995), which is generally regarded as the most reliable He–He potential. V_{33} is given by

$$V_{33}(r) = \varepsilon A \exp(-\beta x) - \left(\frac{C_6}{x^6} + \frac{C_8}{x^8} + \frac{C_{10}}{x^{10}}\right) \varepsilon f(x); \qquad (1)$$

where

$$f(x) = \begin{cases} \exp\left[-\left(\dfrac{D}{x} - 1\right)^2\right], & x < D \\ 1, & x \geq D \end{cases}$$

and $x \equiv r/r_m$; $D = 1.24$; $A = 0.554 \times 10^6$; $\beta = 13.35$; $C_6 = 1.373$; $C_8 = 0.425$; $C_{10} = 0.178$; $\varepsilon = 10.8$ K.

The second effect is the induced potential arising from the interaction between the ^{3}He atom with the HeII background through the bare potential V_{34}. This has been derived by Compbell (1967) and is given by

$$V_{34}(r) = -2(1+\alpha)V_{34}([r_{34}])g(r), \tag{2}$$

where

$$g(r) \equiv \begin{cases} \dfrac{3y}{4} - \dfrac{y^3}{16}, & y \leq 2 \\ 1, & y > 2 \end{cases}$$

with

$$y \equiv \left(\frac{4\pi}{3\omega_3}\right)^{-1/3}; \quad \omega_3 = (1+\alpha)\omega_4.$$

Here, ω_3 (ω_4) is the volume per particle occupied by a ^{3}He (HeII) atom, and $[r_{34}] \equiv \max(4.0, r)$. The factor α is the volume differential coefficient. The third effect is associated with the induced potential HeII–HeII background interaction through the bare potential V_{44}. The corresponding term in the effective interaction assumes the form (Compbell, 1967)

$$V_{44}(r) = (1+\alpha)^2 V_{44}([r_{44}])g^2(r), \tag{3}$$

where $[r_{44}] \equiv \max(3.8, r)$. The total effective interatomic potential between two ^{3}He atoms is therefore

$$V(r) = V_{33}(r) + V_{34}(r) + V_{44}(r). \tag{4}$$

The total Hamiltonian operator describing mixtures of ^{3}He quasiparticles embedded in the HeII background at low temperatures in the configuration space based on the Fermi-liquid model (Emery, 1966, 1967), is

$$H = \sum_i \frac{p_i^2}{2m_3^*} + \frac{1}{2}\sum_{i \neq j} V(r_{ij}), \tag{5}$$

where $V(r_{ij})$ is the effective interatomic potential between two ^{3}He atoms, and m_3^* is the effective mass of a ^{3}He atom. At very low temperatures, the ^{3}He atoms are regarded as a gas in a neutral nonviscous medium of constant density. The effect of the superfluid HeII is taken into account through the effective interatomic potential and the effective mass in the kinetic energy term.

In second quantization representation the effective Hamiltonian for ^{3}He atoms is the sum of the kinetic energy term (T) of the fermionic particles and the effective interatomic potential term (V):

$$\hat{H} = \hat{T} + \hat{V}. \tag{6}$$

The Hamiltonian for a fermionic system is (Al-Sugheir and Ghassib, 2002)

$$\hat{H} = \sum_{\vec{k}\lambda} \frac{\hbar^2 k^2}{2m_3^*} \hat{a}^+_{k\lambda} \hat{a}_{k\lambda} + \frac{1}{2\Omega} \sum_{\vec{k}\vec{q}\vec{p},\lambda\lambda_1} V(k) \hat{a}^+_{p-k\lambda} \hat{a}_{p\lambda} \hat{a}^+_{q\lambda_1} \hat{a}_{q-k\lambda_1}, \tag{7}$$

where the sum is over the complete set of single-particle quantum numbers. The index $\vec{k}$ denotes the linear momentum of the particle and the index λ represents the spin of the particle. $\hat{a}^+_{k\lambda}$ and $\hat{a}_{k\lambda}$ are the creation and annihilation operators, and $V(k)$ is the Fourier transform of the effective pair interatomic potential which is defined as

$$V(k) = \int V(r) \exp(i\vec{k}.\vec{r})\, d\vec{r}. \tag{8}$$

3. THE STATIC APPROXIMATIOM FORMALISM FOR DILUTE ^{3}He–HeII MIXTURES

In the static approximation the Hamiltonian is expressed as a linear combination of the local-field operators $\hat{E}_{k\lambda}$:

$$\hat{H} = \sum_{k\lambda} \hat{E}_{k\lambda} \hat{a}^+_{k\lambda} \hat{a}_{k\lambda}, \tag{9}$$

where the local-field operator $\hat{E}_{k\lambda}$ is a function of the creation and annihilation operators and is assumed to be hermitian and that it commutes with the creation and annihilation operators.

To calculate the thermodynamic properties we need the grand Hamiltonian rather than the Hamiltonian, which is defined as

$$\hat{H} = \sum_{k\lambda} (\hat{E}_{k\lambda} - \mu) \hat{a}^+_{k\lambda} \hat{a}_{k\lambda}, \tag{10}$$

where μ is the chemical potential. The Heisenberg representation of the creation operator $\hat{a}^+_{k\lambda}(\tau)$ for a grand Hamiltonian is given by

$$\hat{a}^+_{k\lambda}(\tau) = \exp(\tau \hat{H}) \hat{a}^+_{k\lambda}(0) \exp(-\tau \hat{H}), \tag{11}$$

where $\tau \equiv it$. The local-field operator can be calculated from the equation of motion of the operator $\hat{a}^+_{k\lambda}(\tau)$:

$$\frac{d\hat{a}^+_{k\lambda}(\tau)}{d\tau} = [\hat{H}, \hat{a}^+_{k\lambda}(\tau)]. \tag{12}$$

According to (12) and to the anticommutation relations for a Fermi system the equation of motion can be rewritten in the form:

$$\frac{d\hat{a}^+_{k\lambda}(\tau)}{d\tau} = [\hat{H}, \hat{a}^+_{k\lambda}(\tau)] = (\hat{E}_{k\lambda} - \mu)\hat{a}^+_{k\lambda}(\tau). \tag{13}$$

Equation (13) gives:

$$\hat{a}^+_{k\lambda}(\tau) = \hat{a}^+_{k\lambda}(0)\exp\left((\hat{E}_{k\lambda} - \mu)\tau\right). \tag{14}$$

It is customary in statistical quantum mechanics (Feynman, 1972) to replace τ with $\beta = \frac{1}{k_B T}$, k_B being Boltzmann's constant and T the absolute temperature.

$\hat{E}_{k\lambda}$ can be derived from the equation of motion (13) (Al-Sugheir *et al.*, 2001; Al-Sugheir and Ghassib, 2002) and is given by

$$\begin{aligned}\hat{E}_{k\lambda} &= [\hat{a}_{k\lambda}, [\hat{H}, \hat{a}^+_{k\lambda}]_-]_+ \\ &= \varepsilon(k) + \frac{1}{\Omega}\sum_{\vec{q}\lambda_1}(V(0) - V(\vec{k} - \vec{q})\delta_{\lambda\lambda_1})\hat{a}^+_{q\lambda_1}a_{q\lambda_1};\end{aligned} \tag{15}$$

where $\varepsilon(k) = \frac{\hbar^2 k^2}{2m_3^*}$. Inspection of (15) clearly shows that the local-field operator is spin independent, i.e., $\hat{E}_{k\lambda_1} = \hat{E}_{k\lambda_1} = \hat{E}_k$. This is reasonable since the interaction potential $V(\vec{r}_1 - \vec{r}_2)$ is spin independent, and thus the particle distribution operator must also be spin independent

$$\hat{n}_{q\lambda_1} = \hat{a}^+_{q\lambda_1}\hat{a}_{q\lambda_1} = \hat{a}^+_{q\lambda_2}\hat{a}_{q\lambda_2}.$$

The excitation local-field operator, as measured relative to $\hat{E}_0$, is

$$\hat{E}_k = \varepsilon(k) + \frac{1}{\Omega}\sum_{\vec{q}}(V(\vec{q}) - V(\vec{k} - \vec{q}))\,\hat{n}_q. \tag{16}$$

The well-known mean-field approximation states that the local-field operator can be replaced with its mean value; it assumes that the fluctuations in this operator are negligible. In the SA, the square of the local-field operator $\hat{E}^2_k$ is replaced with the mean value $\langle \hat{E}^2_k \rangle$. To calculate the mean value of the square of the local-field operator it is necessary to derive the so-called long-range equation. For this purpose it is sufficient to write the identity

$$\langle \hat{A}(\beta)\hat{C} \rangle = \langle \hat{C}\hat{A} \rangle = \frac{1}{Q}\mathrm{Tr}(\exp(-\beta\hat{H})\hat{C}\hat{A}), \tag{17}$$

where Q is the grand partition function of the system.

The long-range equation relates the mean value of the operator $\hat{n}_k \equiv \hat{a}^+_{k\lambda}\hat{a}_{k\lambda}$ to the mean value of the local-field operator. According to (17)

$$\langle \hat{a}^+_{k\lambda}(\beta)\hat{a}_{k\lambda}\hat{A}\rangle = \langle \hat{a}_{k\lambda}\hat{A}\hat{a}^+_{k\lambda}\rangle. \tag{18}$$

If the operator $\hat{A}$ is chosen to commute with $\hat{a}^+_{k\lambda}$, $\hat{a}_{k\lambda}$, and $\hat{E}_k$, equation (18) can be written as

$$\langle \hat{a}^+_{k\lambda}(\beta)\hat{a}_{k\lambda}\hat{A}\rangle = \langle \hat{a}_{k\lambda}\hat{A}\hat{a}^+_{k\lambda}\rangle = \langle \hat{a}_{k\lambda}\hat{a}^+_{k\lambda}\hat{A}\rangle. \tag{19}$$

On the basis of this equation we can write

$$\langle \hat{a}_{k\lambda}\hat{a}^+_{k\lambda}\hat{A}\rangle = \langle \hat{a}^+_{k\lambda}(\beta)\hat{a}_{k\lambda}\hat{A}\rangle = \langle \hat{a}^+_{k\lambda}\exp(\beta(\hat{E}_k - \mu))\hat{a}_{k\lambda}\hat{A}\rangle. \tag{20}$$

It is desirable to write the exponential term in (20) as a linear form of the local-field operator; this could be done with the aide of the identity (Al-Sugheir *et al.*, 2001; Al-Sugheir and Ghassib, 2002):

$$B(a + b\hat{E}_k) = B_0(k) + B_1(k)\frac{\hat{E}_k}{\Phi_k}, \tag{21}$$

where

$$B_0(k) \equiv \frac{1}{2}(B(a + b\Phi_k) + B(a - b\Phi_k)); \tag{22a}$$

$$B_1(k) \equiv \frac{1}{2}(B(a + b\Phi_k) - B(a - b\Phi_k)), \tag{22b}$$

and Φ^2_k is the kth eigenvalue of the square of the local-field operator $\hat{E}^2_k$:

$$\hat{E}^2_k \cong \langle \hat{E}^2_k\rangle = \Phi^2_k, \tag{23}$$

According to (21, 22a, 22b) we can rewrite (20) in the simple form:

$$\langle (1 - \hat{n}_k)\hat{A}\rangle = B_0(k)\langle \hat{n}_k\hat{A}\rangle + \frac{B_1(k)}{\Phi_k}\langle \hat{n}_k\hat{E}_k\hat{A}\rangle, \tag{24}$$

where

$$B_0(k) = \frac{1}{2}[\exp(\beta(\Phi_k - \mu)) + \exp(-\beta(\Phi_k + \mu))] \tag{25a}$$

and

$$B_1(k) = \frac{1}{2}[\exp(\beta(\Phi_k - \mu)) - \exp(-\beta(\Phi_k + \mu))]. \tag{25b}$$

If, in (24), $\hat{A} \to \hat{E}_k\hat{A}$, we obtain the additional equation:

$$\langle (1 - \hat{n}_k)\hat{E}_k\hat{A}\rangle = B_0(k)\langle \hat{n}_k\hat{E}_k\hat{A}\rangle + B_1(k)\langle \hat{n}_k\hat{A}\rangle\Phi_k. \tag{26}$$

To obtain the desired long-range equation, $\langle \hat{n}_k \hat{E}_k \hat{A} \rangle$ is eliminated from (24) and (26); so that

$$\langle \hat{n}_k \hat{A} \rangle = \eta_0(k) \langle \hat{A} \rangle + \frac{\eta_1(k)}{\Phi_k} \langle \hat{E}_k \hat{A} \rangle, \tag{27}$$

where

$$\eta_0(k) = \frac{1}{2} \left(\frac{1}{\exp(\beta(\Phi_k - \mu)) + 1} + \frac{1}{\exp(-\beta(\Phi_k + \mu)) + 1} \right) \tag{28a}$$

and

$$\eta_1(k) = \frac{1}{2} \left(\frac{1}{\exp(\beta(\Phi_k - \mu)) + 1} - \frac{1}{\exp(-\beta(\Phi_k + \mu)) + 1} \right). \tag{28b}$$

The necessary quantity to calculate the thermodynamic properties, Φ_k, can be calculated from a closed set of nonlinear integral equations generated from the long-range equation.

4. THE CLOSED SYSTEM OF NONLINEAR INTEGRAL EQUATIONS FOR DILUTE ^{3}He–HeII MIXTURES

The closed system of nonlinear integral equations can be generated from (27). To calculate Φ_k, we chose $\hat{A} = \hat{E}_k$ and substituted this in (27), and obtained

$$\eta_1(k)\Phi_k = \langle \hat{n}_k \hat{E}_k \rangle - \eta_0(k) \langle \hat{E}_k \rangle. \tag{29}$$

The first term in this equation is calculated from (16)

$$\langle \hat{n}_k \hat{E}_k \rangle = \langle \hat{n}_k \rangle \varepsilon(k) + \frac{1}{\Omega} \sum_{\vec{q}} (V(\vec{q}) - V(\vec{k} - \vec{q})) \langle \hat{n}_k \hat{n}_q \rangle. \tag{30}$$

To calculate Φ_k it is necessary to calculate the correlation relation $\langle \hat{n}_k \hat{n}_q \rangle$. To achieve that we consider $\hat{A} = \hat{n}_q$ in (27), where $\vec{q} \neq \vec{k}$ because $\hat{A}$ must commute with the local-field operator and with the creation and annihilation operators:

$$\begin{aligned} \langle \hat{n}_k \hat{n}_q \rangle_{\mathrm{c}} &= \eta_0(k) \langle \hat{n}_q \rangle + \frac{\eta_1(k)}{\Phi_k} \langle \hat{E}_k \hat{n}_q \rangle \\ &= \eta_0(k) \langle \hat{n}_q \rangle + \frac{\eta_1(k)}{\Phi_k} \left(\varepsilon(k) \langle \hat{n}_q \rangle + \frac{1}{\Omega_3} \sum_{\vec{p}} (V(\vec{p}) - V(\vec{P} - \vec{k})) \langle \hat{n}_p \hat{n}_q \rangle \right), \end{aligned} \tag{31}$$

where the subscript "c" means that $\vec{q} \neq \vec{k}$. If $\vec{q} = \vec{k}$ we should start from (18) to calculate $\langle \hat{n}_k^2 \rangle$ and after some manipulation, we have

$$\langle \hat{n}_k^2 \rangle = \zeta_0(k)[1 - 2\langle \hat{n}_k \rangle] + \zeta_1(k)[1 - 2\eta_0(k)] \frac{\langle \hat{E}_k \rangle}{\Phi_k} - 2\zeta_1(k)\eta_1(k); \tag{32}$$

where

$$\zeta_0(k) = \frac{1}{2}\left(\frac{1}{\exp(\beta(\Phi_k - \mu)) - 1} + \frac{1}{\exp(-\beta(\Phi_k + \mu)) - 1}\right), \tag{33a}$$

$$\zeta_1(k) = \frac{1}{2}\left(\frac{1}{\exp(\beta(\Phi_k - \mu)) - 1} - \frac{1}{\exp(-\beta(\Phi_k + \mu)) - 1}\right). \tag{33b}$$

Also, it is necessary to calculate the particles distribution as well as the mean value of the local-field operator. Choosing $\hat{A} = 1$ in (27) we obtain

$$\langle \hat{n}_k \rangle = \eta_0(k) + \frac{\eta_1(k)}{\Phi_k}\langle \hat{E}_k \rangle, \tag{34}$$

and the mean value of the local-field operator is

$$\langle \hat{E}_k \rangle = \varepsilon(k) + \frac{1}{\Omega}\sum_{\vec{q}} (V(\vec{q}) - V(\vec{k} - \vec{q}))\langle \hat{n}_{\vec{q}} \rangle. \tag{35}$$

Finally, the chemical potential of our system, which plays a crucial role in calculating the functions η_0, η_1, ζ_0, and ζ_1, is calculated according the condition

$$\frac{1}{\Omega}\sum_{\vec{k}} \langle \hat{n}_{\vec{k}} \rangle = \rho_3, \tag{36}$$

ρ_3 being the density number of ^{3}He particles in our system.

The set of nonlinear integral equations (29–32, 34, 35) can be solved numerically to calculate the thermodynamic properties of ^{3}He–HeII mixtures.

In the thermodynamic limit the summation in all previous equations can be changed to integration, and we can integrate over the solid angle directly. These equations become

$$\langle \hat{n}_k \hat{E}_k \rangle = \langle \hat{n}_k \rangle \varepsilon(k) + \frac{1}{2\pi^2}\int_0^\infty (V(q,0) - V(k,q))\langle \hat{n}_k \hat{n}_q \rangle q^2\, dq; \tag{37}$$

$$\langle \hat{n}_k \hat{n}_q \rangle_c = \frac{\eta_1(k)}{\Phi_k}\left(\frac{1}{2\pi^2}\int_0^\infty (V(p,0) - V(p,k))\langle \hat{n}_p \hat{n}_q \rangle p^2\, dp\right) + \eta_0(k)\langle \hat{n}_q \rangle + \frac{\eta_1(k)}{\Phi_k}\varepsilon(k)\langle \hat{n}_q \rangle; \tag{38}$$

and

$$\langle \hat{E}_k \rangle = \varepsilon(k) + \frac{1}{2\pi^2}\int_0^\infty (V(q,0) - V(k,q))\langle \hat{n}_q \rangle q^2\, dq. \tag{39}$$

$V(k, p)$ is the Fourier–Bessel transform of the potential and is given by

$$V(k, p) = 4\pi \int_0^\infty \frac{\sin(kr)}{kr}\frac{\sin(pr)}{pr} V(r) r^2\, dr. \tag{40}$$

These integral equations (37–39) are calculated by the Gaussian quadrature method (Ali, 1997; Bishop *et al.*, 1977; Ghassib *et al.*, 1976). Our set of nonlinear integral equations will be solved numerically by an iteration method for a realistic interhelium potential: HFDHE2 (Aziz *et al.*, 1979; Janzen and Aziz, 1995).

As a final step to calculate the thermodynamic properties of our system, we should calculate the grand partition function Q;

$$\begin{aligned} Q &= \mathrm{Tr}\,(\exp(-\beta\hat{H})) \\ &= \sum_{n_p}\left\langle \exp\left(-\beta\sum_{\vec{p}\lambda}(\hat{E}_p-\mu)n_p\right)\right\rangle \\ &= \prod_{\vec{p}\lambda}\sum_{n_p}\langle \exp(-\beta(\hat{E}_p-\mu)n_p)\rangle \\ &= \prod_{\vec{p}\lambda}\langle 1+\exp(-\beta(\hat{E}_p-\mu))\rangle. \end{aligned} \tag{41}$$

All the thermodynamic properties depend on the logarithm of the partition function, so it is more convenient to take the logarithm of the above equation. Also, using the SA, this equation becomes linear in the local-field operator

$$\ln(Q) = \sum_{\vec{p}\lambda}\ln\left(1+\exp(\beta\mu)\left[B_0(p)+B_1(p)\frac{\langle\hat{E}_p\rangle}{\Phi_p}\right]\right), \tag{42}$$

where

$$B_0(p) = \frac{1}{2}[\exp(-\beta\Phi_p)+\exp(\beta\Phi_p)], \tag{43}$$

and

$$B_1(p) = \frac{1}{2}[\exp(-\beta\Phi_p)-\exp(\beta\Phi_p)]. \tag{44}$$

The pressure in statistical mechanics (Huang, 1987; Pathria, 1992) is defined as

$$P = \frac{\ln(Q)}{\beta\Omega}, \tag{45}$$

and the mean internal energy is defined as

$$U = -\left(\frac{\partial\ln(Q)}{\partial\beta}\right)_{z,\Omega}, \tag{46}$$

where $z = \exp(\beta\mu)$ is the fugacity.

5. RESULTS AND DISCUSSION

The thermodynamic properties of the ^{3}He–HeII mixtures system is studied at temperatures below 0.1 K to be consistent with the condition on the Hamiltonian describing this system.

The chemical potential, pressure, and the mean internal energy per unit volume were calculated at different temperatures up to 0.1 K and at different values of the differential volume α and the effective mass m_3^*. We found that all these quantities are temperature independent up to 0.1 K, where the distribution function of the fermionic particles is still almost a step function.

The dependence of the chemical potential, mean internal energy per unit volume, and the pressure on the density number of liquid ^{3}He at two different values of the effective masses $m_3^* = 2.33\,m_3$ and $m_3^* = 2.5\,m_3$ are shown in Figures 1–3,

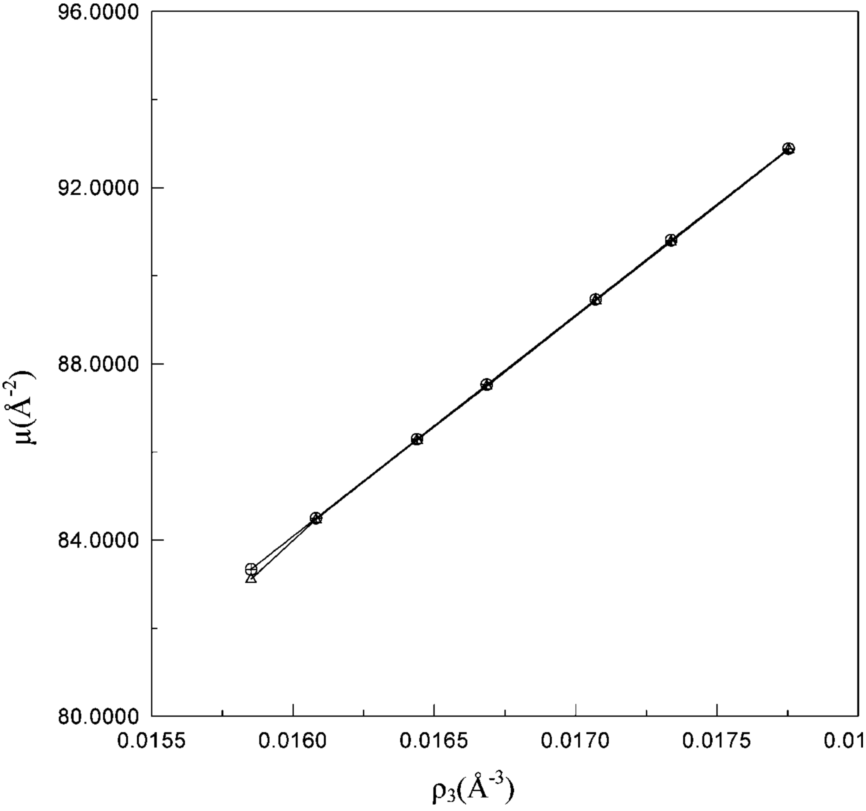

Fig. 1. The chemical potential (μ) for ^{3}He–HeII mixtures as a function of density number (ρ_3). ⊕ for $m_3^* = 2.5m_3$ and (Δ) for $m_3^* = 2.33m_3$.

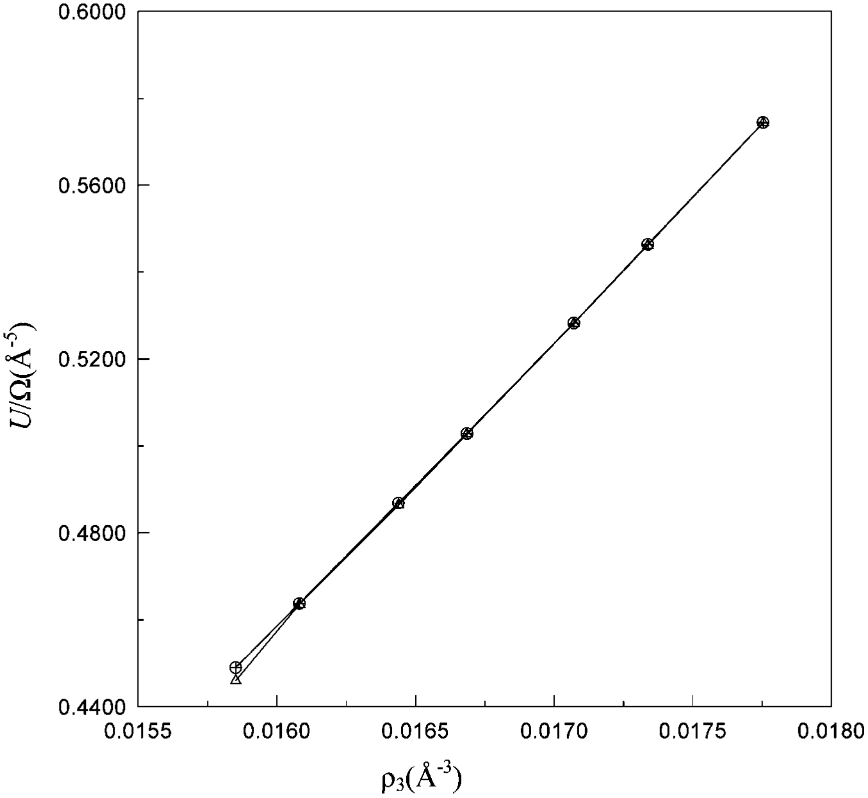

Fig. 2. The mean internal energy per unit volume (U/Ω) for ^{3}He–HeII mixtures as a function of density number (ρ_3). $\oplus$ for $m_3^* = 2.5m_3$ and (Δ) for $m_3^* = 2.33m_3$.

respectively. From these results, we note that the effect of the effective mass is not obvious except at very low-density numbers. This result is not surprising, since the local-field operator is composed of the two terms; the kinetic energy term, which is inversely proportional to the effective mass, and the interaction term, which is independent of the effective mass. The dominant term in the local-field operator is the interaction term, and that explains why the effect of the effective mass was not obvious. At the low-density number limit, the kinetic energy term is appreciable and thus causing the effect of the effective mass to become noticeable. We note that the chemical potential, mean internal energy per unit volume, and the pressure are inversely proportional to the effective mass. This is also expected since the mean internal energy per unit volume is directly proportional to the mean value of the local-field operator, and the other quantities are also proportional to the mean value of the local-field operator but in a complicated form.

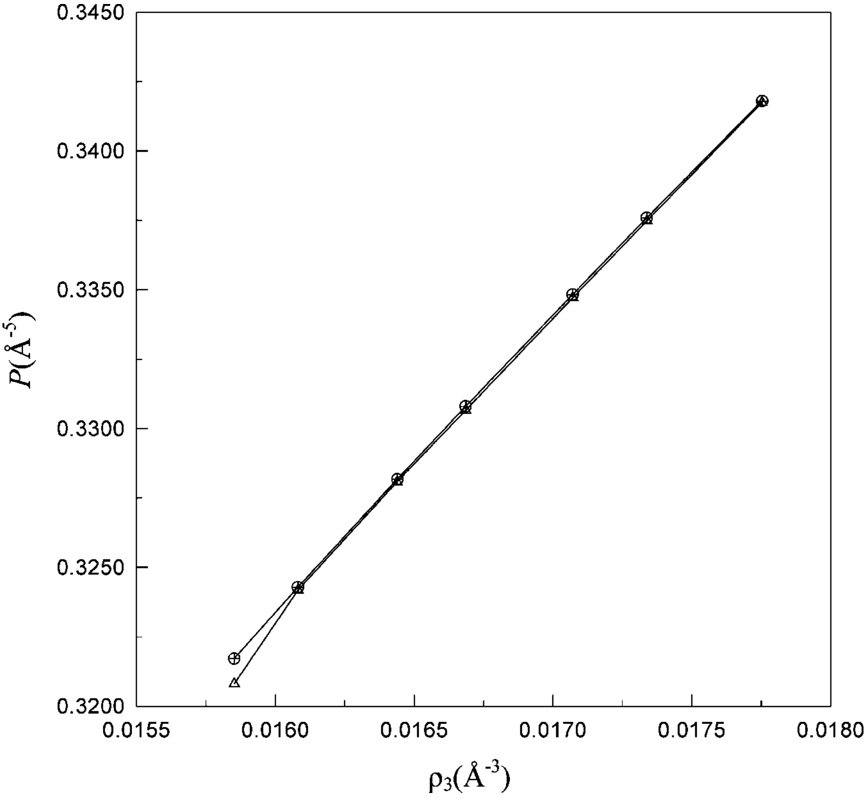

Fig. 3. The pressure (P) for ^{3}He–HeII mixtures as a function of density number (ρ_3). ⊕ for $m_3^* = 2.5m_3$ and (Δ) for $m_3^* = 2.33m_3$.

Figures 1–3 show that the chemical potential, mean internal energy per unit volume, and the pressure are increasing with increasing density number. This is attributed to the increase in chemical potential caused by the occupation of higher states by ^{3}He with increasing density number, and to the increase of the interaction term in the local-field operator; which depends on the number of particles in the occupied states. Theses processes will lead, as a result, to an increase in the mean internal energy and the pressure.

6. CONCLUSION

The static approximation is valid to study ^{3}He–HeII mixtures system. The thermodynamic properties are found to be temperature independent up to 0.1 K, where the distribution function of the fermionic particles has been found to almost be a step function up to 0.1 K. Except at low-density number, the thermodynamic properties are independent of the effective mass where the dominant term in the local-field operator in the occupied states is the interaction term. The superfluid phase in this temperature range is outside the scope of this approximation but may appear if we take into account the higher order approximations.

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